Numerical Solution of Transcendental and Polynomial Equations

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Dr. Sujata Mondal
Assistant professor,
Department of Mathematics,
Mankar College

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Learning outcomes

After studying this chapter, students will be able to

- Understand the polynomial and transcendental equations.
- Know the necessity of applying numerical methods to solve such equations.
- Know about the iterative methods for solving such equations.
- Choose an appropriate method of solving equations.
- Can compare the advantages and disadvantages of various numerical methods.
- Solve polynomial and transcendental equations numerically.

Introduction

If $f(x)$ be a polynomial of n^{th} degree, i.e.,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots.. + a_{n-1}x + a_n \text{ , } a_0 \neq 0$$

Then the equation $f(x) = 0$ is called a polynomial equation of degree n. These types of equations contain only algebraic terms.

The equation $f(x) = 0$ will be called a transcendental equation, if it contains trigonometric, logarithmic, or exponential functions. For example, $x^3 + 2tanx + e^x = 0$ is a transcendental equation.

If $f(x) = 0$ is an algebraic equation of degree less than or equal to 4, direct methods for finding the roots of such equations are available. E.g., a cubic equation can be solved using Cardan's method or a biquadratic equation can be solved using Ferrari's method. But if $f(x)$ is of higher degree or it involves transcendental functions, direct methods do not exist, and we need to apply numerical methods to find the roots of the equation $f(x) = 0$.

Basic concepts and definitions

Convergence

For any iterative numerical method, each successive iteration gives an approximation that moves progressively closer to actual solution, which is known as convergence.

If α be the true value of a root of the equation $f(x) = 0$ and $\{x_n\}$ be the sequence of successive approximation of this root, then the error $\varepsilon_n$at the n-th iteration is defined as

$$\varepsilon_n = \alpha - x_n$$

Now, if we define h_n by

$$h_n = x_{n+1} - x_n$$

then,

$$h_n = x_{n+1} - x_n = (\alpha - \varepsilon_{n+1}) - (\alpha - \varepsilon_n) = \varepsilon_n - \varepsilon_{n+1}$$

and it may be considered as an approximation of ε_n.

The iteration process converges if $\varepsilon_n \to 0$ as $n \to \infty$.

Intermediate Value Theorem

Intermediate Value Theorem states that- if f (x) be continuous function in the closed interval [a,b] and c be any number such that f(a) ≤ c ≤ f(b), then there is at least one number α in [a,b] such that f (α) = c.

Bolzano's Theorem

If $f(x)$ be continuous in the closed interval $[a,b]$ and $f(a)$, $f(b)$ are of opposite signs, then there exists a number α in $[a,b]$ such that $f(\alpha) = 0$, that is, the equation $f(x) = 0$ has a root α in that interval [a, b].

Numerical methods to find roots of algebraic and transcendental equations

The numerical solution of such polynomial or transcendental equations consists of two steps:

Initial guess: Find the smallest possible intervals [a, b] containing one and only one root of the equation f(x) = 0 and take a point $x_0 \in$ [a, b] as an approximation or initial guess to the root of this equation.

Improving the value of the root: If this initial guess x_0 is not in desired accuracy, then it must be improved employing a numerical method. This process of improving the value of a root repeatedly to get the desired accuracy is called the iterative process and such methods are called iterative methods.

Initial guess

Increment Search Method

Most commonly, we use increment search method or tabulation method. This method is used when we need to find an interval where a root of an equation is supposed to be existed. It starts with an initial value x_0 and a small interval Δx. We proceed further to the next as

$$x_1 = x_0 + \Delta x$$

$$x_2 = x_1 + \Delta x$$

$$\ldots\ldots$$

And finally,

$$x_{n+1} = x_n + \Delta x$$

We tabulate the values of x and the corresponding values of $f(x)$. We stop if $f(x_{n+1}).f(x) < 0$ and assure the existence of the root between $[x_{n+1}, x_n]$.

Graphical method

we can also make an initial guess using graphical method. First, we draw the graph of the curve $y = f(x)$ and found the point of intersection of this curve with the x-axis. Any point in the neighborhood of this point may be taken as the initial approximation.

Improving the value of the root

To improve the value of the root, we adopt various numerical methods like –

1. Bisection method (or Bolzano Method)
2. Regula- Falsi Method
3. Newton-Raphson Method
4. Fixed-point iteration
5. Secant method

Bisection Method (or Bolzano Method)

This method is used to find an approximate value of the root of an equation in an interval by repeatedly bisecting it into subintervals.

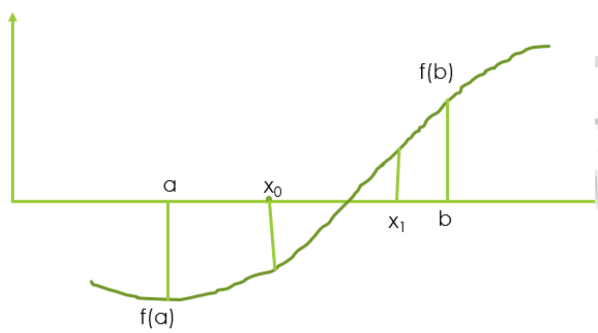

Fig. Pictorial representation of Bisection metho.

Let $f(x)$ be a continuous function in the interval $[a, b]$, such that $f(a)$ and $f(b)$ are of opposite signs, i.e. $f(a) .f(b)$ <0. Take the initial approximation given by $x_0 = \frac{(a+b)}{2}$, one of the three conditions arises for finding the 1st approximation x_1

i. $f(x_0) = 0$, we have a root at x_0.

ii. If $f(a) . f(x_0) < 0$, the root lies between a and x_0 $\therefore$ $x_1 = \frac{a+x_0}{2}$ and repeat the procedure by halving the interval again. We rename the new interval $[a, x_0]$ as $[a_1, b_1]$

iii. If $f(b) . f(x_0) < 0$, the root lies between x_0 and b $\therefore$ $x_1 = \frac{x_0+b}{2}$ and repeat the procedure by halving the interval again. In this case we rename the interval $[x_0, b]$ as $[a_1, b_1]$.

Continue the process until root is found to be of desired accuracy

Example 1. Find a root of the equation $x^4 + 2x^3 - x - 1 = 0$ using bisection method.

Solution: Let $f(x) = x^4 + 2x^3 - x - 1$.

Here, $f(0) = -1$ and $f(1) = 1$. $f(0). f(1) = -1 < 0$. Since f(x) is continuous in [0,1] atleast on root must lie in this interval.

Let $a = 0$, $b = 1$. Then, $x_0 = \frac{0+1}{2} = 0.5$.

Now, $f(0.5) = -1.1875$ and $f(0.5). f(1) < 0$. Therefore, we tale the first approximation as $a = 0.5, b = 1$.

Next, $x_1 = \frac{0.5+1}{2} = 0.75$ and $f(0.75) = -0.5898$. So, $f(0.75). f(1) < 0$, therefore, the next subinterval is taken as $x_2 = \frac{0.75+1}{2} = 0.875$. Here, $f(0.875) = 0.051$ and $f(0.75). f(0.875) < 0$. Therefore, the next subinterval is taken as [0.75, 0.875].

We repeat this process until we get the desired accuracy. After 7th iteration, we get the approximate value of the root as 0.8633 correct up to two decimal places.

Regula- Falsi Method

Regula-Falsi method is also known as method of false position as false position of curve is taken as initial approximation. Let $y=f(x)$ be represented by the curve PQ. The false position of curve PQ is taken as chord PQ and initial approximation x_0 is the point of intersection of chord PQ with x-axis. Successive approximations x_1, x_2,…are given by point of intersection of chord with $x-$ axis, until the root is found to be of desired accuracy.

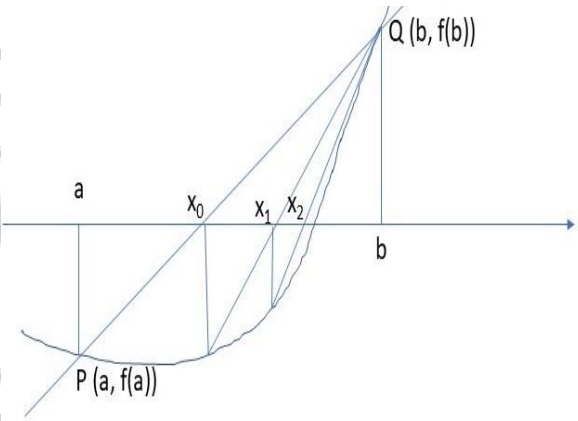

Fig. Pictorial representation of Regula-Falsi method

The equation of the chord PQ is given by

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

If this line cuts the x axis at $(x_0, 0)$, then we get,

$$-f(a) = \frac{f(b) - f(a)}{b - a}(x_0 - a)$$

Simplifying we get,

$$x_0 = \frac{af(b) - bf(a)}{f(b) - f(a)} \dots\dots\dots (1)$$

Now, if $f(x_0)$ is positive, we replace it by b otherwise, we replace it by a and applying the formula (1) we get the successive approximate values of x as $x_1, x_2, \dots \dots \dots, x_n$.

Example 2: Find a real root of the equation $x\, log_{10}(x) - 1.2 = 0$ using Regula-Falsi method.

Solution: Let $f(x) = x\, log_{10}(x) - 1.2$.

Here, $f(2) = -0.6$ and $f(3) = 0.23$, $f(2).f(3) < 0$. Therefore, a root must lie in [2, 3].

n	a	b	f(a)	f(b)	$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x_n)$
0	2	3	− 0.59794	0.231364	2.72101	− 0.0170911
1	2.72101	3	− 0.0170911	0.231364	2.74021	− 0.000384056
2	2.74021	3	− 0.000384056	0.231364	2.74064	− 8.58134E-6
3	2.74064	3	− 8.58134E-6	0.231364	2.74065	− 1.91717E-7

Therefore, the root of this equation is 2.7406, correct up to four decimal places.

Newton-Raphson Method

If x_0 be the approximate value of the root α and h be the error in x_0, then $\alpha = x_0 + h$. Then,

$$0 = f(\alpha) = f(x_0 + h) = f(x_0) + hf'(x_0) \Rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

Therefore, the first approximation is,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The next estimation x_2 is obtained from x_1 in the same way as x_1 was obtained from x_0, i.e.,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Continuing in this way we have, if x_n is the current estimate, then the next estimate x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Note: Sometimes the following two cases may occur.

Case I: If any of the approximations encounters a zero derivative (extreme point), then the tangent at that point goes parallel to x-axis, resulting in no further approximations.

Case II: Sometimes Newton-Raphson method may run into an infinite cycle or loop. Change in initial approximation may untangle the problem.

Remarks

- Newton-Raphson method can be used for solving both algebraic and transcendental equations and it can also be used when roots are complex.
- Initial approximation x_0 can be taken randomly in the interval $[a, b]$, such that $f(a) .f(b) <0$
- Newton-Raphson method has quadratic convergence, but in case of bad choice of x_0 (the initial guess), Newton- Raphson method may fail to converge
- This method is useful in case of large value of $f'(x_n)$ i.e. when graph of $f(x)$ while crossing x -axis is nearly vertical.

Example 3: Use Newton-Raphson method to find a root of the equation $x^3-5x+3=0$ correct to three decimal places.

Solution: Let, $f(x) = x^3 - 5x + 3$

Then, $f'(x) = 3x^2 - 5$

Here, $f(0) = 3$ and $f(1) = -1 \Rightarrow f(0) .f(1) <0$

Also, $f(x)$ is continuous in [0, 1], $\therefore$ at least one root exists in [0, 1]

Let initial approximation x_0 in the interval [0, 1] be 0.8 then,

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where $x_0=0.8$, $f(0.8) = -0.488$, $f'(0.8) = -3.08$

$\Rightarrow x_1 = 0.8 - \frac{-0.488}{-3.08} = 0.6416$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, where x_1=0.6415, $f(0.6416)$ =0.0561, $f'(0.6416) = -3.7650$ ⇒x_2=0.6416− $\frac{0.0561}{-3.7650}$ =0.6565

Again, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$ where x_2=0.6565, $f(0.6565)$ =0.0004, f' (0.6565) = −3.7070

⇒x_3=0.6565− $\frac{0.0004}{-3.7070}$ =0.6566

Hence, a root of the equation x^3−5x+3=0 correct to three decimal places is 0.6566

Fixed-Point Iteration Method

Let $[a_0, b_0]$ be an initial small interval containing the only root α of the given equation $f(x)$ = 0. We can rewrite the given equation as, $x = \phi(x)$.

Let x_0 be an approximation to the desired root, which we can be found graphically or otherwise. Substituting x_0 in right hand side of (1), we get the first approximation as

$x_1 = \phi(x_0)$

In the same way we can get the successive approximations as

$x_2 = \phi(x_1)$, $x_3 = \phi(x_2)$ ….. $X_{n+1} = \phi(x_n)$

Condition of convergence of Fixed-Point Iteration method

Let α be the root of the equation f (x) = 0, that is, x = α be a solution of x = ϕ(x) and suppose ϕ(x) has a continuous derivative in some interval [a0, b0] , containing the root α. If $|\phi'(x)| \leq K < 1$ for all x in [a0, b0] , then the fixed-point iteration process xn+1 = ϕ(xn) converges with any initial approximation x0 in [a0, b0].

Example 4. Find a positive root of the equation $xe^x = 1$, using Fixed-Point Iteration method.

Solution: We can rewrite the given equation as $x = e^{-x}$.

Let $\phi(x) = e^{-x}$. Here, $|\phi'(x)| < 1$ for x < 1, therefore, it is possible to apply Fixed-Point Iteration method.

Let, $x_0 = 1$, then,

$$x_1 = e^{-1} = 0.3678794$$

$$x_2 = e^{-0.3678794} = 0.6922006$$

$$x_3 = e^{-0.6922006} = 0.5004735$$

Proceeding in this way we can get the desired root as x = 0.5671.

Secant Method

In Newton-Raphson method, sometimes the computation of derivative of function may be difficult. So, in secant method, the derivative at x_n is approximated by the following difference quotient:

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Hence, from the iteration scheme of the Newton-Raphson method we have –

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})f(x_n)}{f(x_n) - f(x_{n-1})}$$

Geometrical significance of Secant Method

The equation of the straight line joining the points (x_n, f(x_n)), (x_{n-1}, f(x_{n-1})) is given by

$$\frac{y - f(x_n)}{x - x_n} = \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

Suppose it cuts the x axis at (x_{n+1}, 0). Then,

$$x_{n+1} = x_n - \frac{(x_{n-1} - x_n)f(x_n)}{f(x_{n-1}) - f(x_n)}$$

Therefore, in this method we approximate the curve y = f(x) between (x_n, f(x_n)), (x_{n-1}, f(x_{n-1})) by a straight line.

Example 5. Find the root of the equation $10^x + x - 4 = 0$ using secant method

Order of convergence

Any numerical method is said have order of convergence ρ, if ρ is the largest positive number such that $\left|\frac{\epsilon_{n+1}}{\epsilon_n^{\rho}}\right| \leq k$, where ϵ_n and ϵ_{n+1} are errors in nth and $(n+1)th$ iterations, k is a finite positive constant.

Order of convergence of Bisection Method

Here,

$$b_n - a_n = \frac{(b_{n-1} - a_{n-1})}{2} = \frac{1}{2}.\frac{b_{n-2} - a_{n-2}}{2} = \frac{1}{2^2}(b_{n-2} - a_{n-2}) = \frac{1}{2^3}(b_{n-3} - a_{n-3}) \dots.$$

$$= \frac{b-a}{2^n} \dots\dots \quad (1)$$

Now,

$$|\varepsilon_n| = |\alpha - x_n| \leq |b_n - a_n| = \frac{b-a}{2^n}$$

Therefore,

$$|\varepsilon_{n+1}| \leq \frac{b-a}{2^{n+1}}$$

That implies,

$$\left|\frac{\varepsilon_{n+1}}{\varepsilon_n}\right| \cong \frac{1}{2}$$

Hence, the order of convergence for bisection method is 1.

Order of convergence of Fixed-Point Iteration Method

If α be a root of the equation $f(x) = 0$, i.e. of $x = \phi(x)$, then,

$$\alpha = \phi(\alpha) \dots\dots\dots (1)$$

Again, according to this method,

$$x_{n+1} = \phi(x_n) \dots\dots\dots (2)$$

From (1) and (2), we have-

$\alpha - x_{n+1} = \phi(\alpha) - \phi(x_n) = (\alpha - x_n)\phi'(\xi)$ Where, $\min(\alpha, xn) < \xi < \max(\alpha, x_n)$

Therefore, $|\varepsilon_{n+1}| = |\varepsilon_n|\phi'(\xi)$, i.e. $\left|\frac{\varepsilon_{n+1}}{\varepsilon_n}\right| = \phi'(\xi) = k$(say)

Hence, this method converges linearly.

Hands on

Q 1. Find a real root of equation $x^3 - x - 11 = 0$ by Bisection method.

Q 2. Find a real root of $x^3 - 5x + 3 = 0$ using Bisection method.

Q.3. Use Regula-Falsi method to find a root of the equation $x\log_{10}x - 1.2 = 0$ correct to two decimal places.

Q.4. Find the equation of the root $x^3 = 1 - x^2$ in the interval [0,1] using iteration method.

Q.5. Find a real root of the equation $xe^x - 2 = 0$ correct upto five decimal places.

References

1. Sastry S. Introductory methods of numerical analysis / S.S. Sastry. In 1984.

2. Ray SS. Numerical Analysis with Algorithms and Programming [Internet]. CRC Press; 2016. Available from: https://books.google.co.in/books?id=Nu0bDAAAQBAJ